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Calculation of Electromagnetic Scattering by a Perfect Conductor*

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A new method is presented for calculating the scattering of an arbitrary electromagnetic wave by a bounded, perfectly conducting body of general shape. The strategy is to replace the corresponding exterior boundary-value problem by an approximate problem on the boundary of the scattering body. This involves the introduction of a certain bilinear form and non-local boundary operator, together with the use of a special class of known solutions of the reduced Maxwell's equations satisfying the Sommerfeld radiation boundary conditions at infinity. Two computer programs implementing this method are described and numerical results showing the successful application of this method to some model problems are presented.

1. INTRODUCTION

The numerical calculation of scattering by an arbitrarily shaped perfect conductor or dielectric body has received much attention in recent years. The problem is important in many areas including the design of missile fuses, the assessment of damage by an electromagnetic pulse, and the study of the biological effects of microwave radiation. Mathematically, the problem has the form of an exterior boundary-value problem for a system of elliptic partial differential equations. The problem has particular difficulties arising from the large number of unknowns, the unbounded domain, and the fact that the solution is oscillatory for large values of frequency.

The principal techniques for the numerical solution of scattering problems are the

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method of integral equations, and the finite-difference or finite-element method. We shall briefly discuss these methods.

In the method of integral equations, one starts with a formulation of the boundary-value problem as an integral equation on the surface of the scattering body; the unknown may, e.g., be the current density on the surface. An excellent survey of such formulations, in the case of the reduced wave equation, is given in [1]. The integral equation is solved numerically, either by using a collocation method or a Galerkin method (method of moments) to reduce the problem to a finite system of linear equations. The principal advantage of the method is that the dimension of the problem has been reduced; one must find a vector field on the two-dimensional surface of the scattering body, instead of in the three-dimensional exterior region. (This advantage is lost in the case of inhomogeneous penetrable bodies; see, e.g., [3].) On the other hand, the system of linear equations has a full (non-sparse) coefficient matrix. Also, the integral equation contains a weakly singular kernel that comes from the fundamental solution of the problem; the resulting surface integrals may be difficult to evaluate with sufficient accuracy. Examples of the integral equation approach are contained in [2, 3]. In particular, [2] contains an exposition of the Galerkin method for these problems.

With the second method, the boundary condition at infinity is replaced by a boundary condition on the surface of a large sphere containing the scattering body. This approach is used in [4]. The resulting boundary-value problem is discretized by means of a finite-difference or finite-element method. This method has the advantage of producing a simple, sparse, coefficient matrix. On the other hand, the order of the matrix will be larger, because the unknown is now a vector field in a three-dimensional domain. A judicious use of graded meshes at a large distance from the scattering body will help alleviate this problem [5]. Examples of the use of finite differences or finite elements are contained in [6–8]. In [8], the problem of a matrix of large order is overcome through the use of an iterative scheme for the solution of the linear system.

The finite-element method is especially appropriate for the problem of penetration of electromagnetic fields into an inhomogeneous absorbing body. To treat such problems, one couples the finite-element procedure inside the body with an integral equation or other technique on the surface of the absorbing body. For some work on this, see [9–11]. A particularly successful coupling technique has been developed by Waterman [12, 13]. In Waterman's method, the Stratton–Chu formula is used to represent the scattered electric field $\mathbf{E}^1$ in terms of integrals of the tangential components $\mathbf{n} \times \mathbf{E}^1$ and $\mathbf{n} \times \mathbf{H}^1$ of the electric and magnetic fields over the scattering surface. The free-space Green's function enters into this representation. The electromagnetic fields and Green's function are expanded in a series of harmonic vector fields, and after truncating the series, a set of linear equations is obtained for the unknown coefficients in the expansion. This eliminates the singularity from the kernel of the equation and gives a representation of the approximate solution in terms of fields which already satisfy the differential equations of the problem. It should be noted, however, that although the resulting integrands no longer contain singularities,

they contain oscillating terms, and care in their evaluation is still required. Examples of further work along this line are contained in [14–16].

The essence of the method proposed here may be described as follows: Let $\mathbf{E}^1$ and $\mathbf{H}^1$, respectively, be the electric and magnetic fields of the scattered wave, let $\mathbf{E}^0$ and $\mathbf{H}^0$ represent the incoming electromagnetic wave, and let $\mathbf{u}$ and $\mathbf{v}$ represent any electromagnetic wave which satisfies the outgoing radiation conditions. Since $\mathbf{n} \times \mathbf{E}^1 = -\mathbf{n} \times \mathbf{E}^0$ on the boundary Γ of the scattering body

$$\oint_\Gamma \mathbf{n} \times \mathbf{E}^1 \cdot \mathbf{v} d\Gamma = -\oint_\Gamma \mathbf{n} \times \mathbf{E}^0 \cdot \mathbf{v} d\Gamma. \tag{1.1}$$

Equation (1.1) may be regarded as a way of imposing the boundary conditions using Galerkin's method. If $\mathbf{E}^1$, $\mathbf{H}^1$ and $\mathbf{u}$, $\mathbf{v}$ are restricted to a finite-dimensional collection of electromagnetic fields, the resulting finite system of linear equations given by (1.1) provides the approximate solution to the problem that is considered here.

Our method has similarities to the method of Waterman, since we also use harmonic vector fields. Our formulation is different, however, since the Stratton–Chu formula is not used. We view the problem as a system of differential equations coupled with a complicated non-local boundary condition. A similar point of view is taken in [17]. The principal difference between our approach and that of [17] is, in [17], which treats a two-dimensional problem, the non-local boundary condition is imposed on an auxiliary curve enclosing the scattering body. In our approach, the nonlocal boundary condition is imposed on the scattering body itself. This avoids having to introduce an extra finite-element procedure to approximate the field in the region between the scattering body and the auxiliary curve.

The method presented here was first developed for the penetration problem [18] and was described in a preliminary way in [19, 20].

2. Mathematical Formulation

Let Ω be a bounded, perfectly conducting, three-dimensional domain with boundary Γ. We allow the possibility that Ω is a disconnected domain, in which case Γ is a collection of disjoint closed surfaces. Let Ω_0 be the set of points not in Ω. We are given an incident electromagnetic wave $\mathbf{E}^0$, $\mathbf{H}^0$ of frequency ω. Thus, taking out a time-harmonic factor of the form $e^{-i\omega t}$, $\mathbf{E}^0$ and $\mathbf{H}^0$ are vector fields defined in all space, which satisfy the reduced Maxwell's equations

$$\nabla \times \mathbf{E}^0 = \alpha \mathbf{H}^0, \qquad \nabla \times \mathbf{H}^0 = \beta \mathbf{E}^0.$$

If the wave is propagating in a vacuum, $\alpha = i\omega\mu_0$ and $\beta = -i\omega\varepsilon_0$, where $\varepsilon_0 = 8.854 \times 10^{-12}$F/m, $\mu_0 = 1.257 \times 10^{-6}$ H/m. In general, $\alpha = i\alpha_0$ and $\beta = -i\beta_0$, where α_0, β_0 are positive real numbers. Let $\mathbf{E}$, $\mathbf{H}$ be the electromagnetic wave resulting from the scattering of the incident wave by the perfectly conducting body Ω.

Let $\mathbf{E}^1 = \mathbf{E} - \mathbf{E}^0$, $\mathbf{H}^1 = \mathbf{H} - \mathbf{H}^0$ be the scattered wave. Then, $\mathbf{E}(x)$, $\mathbf{H}(x)$ are defined for $x \in \Omega_0$, and are determined by the set of equations[1]

$$\nabla \times \mathbf{E} = \alpha \mathbf{H}, \qquad x \in \Omega_0 \tag{2.1a}$$

$$\nabla \times \mathbf{H} = \beta \mathbf{E}, \qquad x \in \Omega_0 \tag{2.1b}$$

$$\mathbf{E} \times \mathbf{n} = 0, \qquad x \in \Gamma \tag{2.2}$$

$$\mathbf{E}^1, \mathbf{H}^1 = O(r^{-1}), \qquad r \to \infty \tag{2.3a}$$

$$\mathbf{e}_r \times \mathbf{E}^1 - \sqrt{\alpha_0/\beta_0}\, \mathbf{H}^1 = o(r^{-1}), \qquad r \to \infty \tag{2.3b}$$

$$\mathbf{e}_r \times \mathbf{H}^1 + \sqrt{\beta_0/\alpha_0}\, \mathbf{E}^1 = o(r^{-1}), \qquad r \to \infty. \tag{2.3c}$$

Here, $\mathbf{n}$ denotes the outward pointing unit normal to Γ, $r = |x|$, and $\mathbf{e}_r$ denotes the unit vector in the radial direction.

System (2.1a, b) comprises a system of six partial differential equations in the six unknowns consisting of the components of $\mathbf{E}$ and $\mathbf{H}$. Equations (2.1–2.3) form an exterior boundary-value problem for this system; (2.2) is the boundary condition on Γ, and (2.3) are the boundary conditions at infinity. This problem has been treated, e.g., in [21], where it is shown that, for reasonable surfaces Γ and incident waves $\mathbf{E}^0$, $\mathbf{H}^0$, the problem has a unique solution.

It is convenient to discuss the boundary-value problem in terms of the scattered wave. For this, let $\mathbf{f}$ be a tangential vector field on Γ, and consider the boundary–value problem defined by

$$\nabla \times \mathbf{E}^1 = \alpha \mathbf{H}^1, \qquad x \in \Omega_0, \tag{2.4a}$$

$$\nabla \times \mathbf{H}^1 = \beta \mathbf{E}^1, \qquad x \in \Omega_0, \tag{2.4b}$$

$$\mathbf{n} \times \mathbf{E}^1 = \mathbf{f}, \qquad x \in \Gamma, \tag{2.5}$$

and (2.3). From [21], it is known that this problem has a unique solution. If $\mathbf{f} = -\mathbf{n} \times \mathbf{E}^0$, then the solution $\mathbf{E}^1$, $\mathbf{H}^1$ of (2.3)–(2.5) gives the scattered wave for the original problem. Thus, it suffices to solve (2.3)–(2.5).

Let $\mathbf{f}$ be a tangential vector field on Γ. We define another tangential field $K\mathbf{f}$ as follows. Let $\mathbf{E}^1$, $\mathbf{H}^1$ be the solution of (2.3–2.5), and let $K\mathbf{f} = n \times \mathbf{H}^1$. The operator K maps tangential vector fields into tangential vector fields. We also define a bilinear form B on tangential vector fields. If $\mathbf{f}$ and $\mathbf{g}$ are two tangential vector fields on Γ, we define

$$B(\mathbf{f}, \mathbf{g}) = \oint_\Gamma \mathbf{n} \cdot \mathbf{f} \times K\mathbf{g} d\Gamma. \tag{2.6}$$

The operator K and the bilinear form B are used in the formulation of our numerical method. We prove that the bilinear form is symmetric.

[1] We use here and in the sequel, the notation $\phi(r) = O(\psi(r))$, $r \to \infty$, to mean $|\phi(r)/\psi(r)| \leqslant C$, $r \to \infty$. and $\phi(r) = o(\psi(r))$, $r \to \infty$, to mean $\phi(r)/\psi(r) \to 0$, $r \to \infty$.

LEMMA 1. $B(\mathbf{f}, \mathbf{g}) = B(\mathbf{g}, \mathbf{f})$.

Proof. Let $\mathbf{E}^1$, $\mathbf{H}^1$ be the solution of (2.3)–(2.5), and let $\mathbf{E}^2$, $\mathbf{H}^2$ be the solution of (2.3)–(2.5) with $\mathbf{f}$ replaced by $\mathbf{g}$. Let B_r be a ball of radius r with boundary S_r. Let r be chosen so large that $\Omega \subset B_r$, and let $\Omega_{0,r} = \Omega_0 \cap B_r$. We have from (2.4a, b)

$$\beta \mathbf{E}^1 \cdot \mathbf{E}^2 - \alpha \mathbf{H}^1 \cdot \mathbf{H}^2 = \nabla \cdot (\mathbf{H}^1 \times \mathbf{E}^2). \tag{2.7}$$

Integrating this over $\Omega_{0,r}$, we obtain

$$-\oint_\Gamma \mathbf{n} \cdot \mathbf{H}^1 \times \mathbf{E}^2 d\Gamma + \oint_{S_r} \mathbf{e}^r \cdot \mathbf{H}^1 \times \mathbf{E}^2 d\Gamma$$

$$= \int_{\Omega_{0,r}} [\beta \mathbf{E}^1 \cdot \mathbf{E}^2 - \alpha \mathbf{H}^1 \cdot \mathbf{H}^2]\, dx. \tag{2.8}$$

The first term on the left side is

$$\oint_\Gamma \mathbf{n} \cdot \mathbf{E}^2 \times \mathbf{H}^1 d\Gamma = \oint_\Gamma \mathbf{n} \cdot \mathbf{g} \times K\mathbf{f} d\Gamma = B(\mathbf{g}, \mathbf{f}).$$

The second term on the left side is

$$\oint_{S_r} \mathbf{E}^2 \cdot \mathbf{e}_r \times \mathbf{H}^1 d\Gamma = -\sqrt{\beta_0/\alpha_0} \oint_{S_r} \mathbf{E}^2 \cdot \mathbf{E}^1 d\Gamma + \delta(r), \tag{2.9}$$

where, using (2.3a, c),

$$|\delta(r)| \leqslant 4\pi r^2 \cdot O(r^{-1}) \cdot o(r^{-1}) \to 0, \qquad r \to 0. \tag{2.10}$$

Since $B(\mathbf{g}, \mathbf{f})$ is independent of r, it follows that the expression

$$\int_{\Omega_{0,r}} (\beta \cdot \mathbf{E}^2 - \alpha \mathbf{H}^1 \cdot \mathbf{H}^2)\, dx + \sqrt{\beta_0/\alpha_0} \oint_{S_r} \mathbf{E}^1 \cdot \mathbf{E}^2\, d\Gamma \tag{2.11}$$

has a finite limit as $r \to \infty$, and that

$$B(\mathbf{g}, \mathbf{f}) = \lim_{r \to \infty} \left\{ \int_{\Omega_{0,r}} (\beta \mathbf{E}^1 \cdot \mathbf{E}^2 - \alpha \mathbf{H}^1 \cdot \mathbf{H}^2)\, dx + \sqrt{\beta_0/\alpha_0} \oint_{S_r} \mathbf{E}^1 \cdot \mathbf{E}^2\, d\Gamma \right\}. \tag{2.12}$$

Since the right side of (2.12) is unchanged if $\mathbf{E}^1$, $\mathbf{H}^1$ and $\mathbf{E}^2$, $\mathbf{H}^2$ are interchanged, the left side of (2.12) is unchanged if $\mathbf{f}$ and $\mathbf{g}$ are interchanged, and the lemma is proved.

The next lemma will be used in the next section to show that the finite system of equations which our method produces always has a solution. To state the result, we let $\bar{z}$ denote the complex conjugate of a complex number z.

LEMMA 2. *If* $B(\mathbf{f}, \bar{\mathbf{f}}) = 0$, *then* $\mathbf{f} = 0$.

Proof. Set $\mathbf{g} = \bar{\mathbf{f}}$ in (2.12) and note that α and β are imaginary. Then

$$\operatorname{Re} B(\mathbf{f}, \bar{\mathbf{f}}) = (\beta_0/\alpha_0)^{1/2} \lim_{r\to\infty} \oint_{S_r} |\mathbf{E}^1|^2 \, d\Gamma. \tag{2.13}$$

In particular, we find that the limit on the right side of (2.13) exists. Suppose $B(\mathbf{f}, \bar{\mathbf{f}}) = 0$. Then

$$\lim_{r\to\infty} \oint_{S_r} |\mathbf{E}^1|^2 \, d\Gamma = 0. \tag{2.14}$$

It follows from a theorem of Rellich [21, Theorem 15] that $\mathbf{E}^1 \equiv 0$. Hence, $\mathbf{f} = 0$.

3. THE APPROXIMATION SCHEME

We describe our numerical method in terms of the scattered wave $\mathbf{E}^1$. Let $\mathbf{g}$ be any tangential vector field on Γ. Then, since $\mathbf{n} \times \mathbf{E}^1 = -\mathbf{n} \times \mathbf{E}^0$ on Γ, we have

$$\mathbf{n} \cdot \mathbf{E}^1 \times K\mathbf{g} = \mathbf{n} \times \mathbf{E}^1 \cdot K\mathbf{g} = -\mathbf{n} \times \mathbf{E}^0 \cdot K\mathbf{g} = -\mathbf{n} \cdot \mathbf{E}^0 \times K\mathbf{g}.$$

Integrating this over Γ, and letting $\mathbf{f}$ denote the tangential component of $\mathbf{E}^1$ on Γ, we obtain

$$B(\mathbf{f}, \mathbf{g}) = -\oint_\Gamma \mathbf{n} \cdot \mathbf{E}^0 \times K\mathbf{g} \, d\Gamma.$$

This equation is the basis for our numerical method. We pick a finite-dimensional collection $\mathscr{S}$ of tangential vector fields on Γ, and we define our approximate solution $\mathbf{f} \in \mathscr{S}$ by

$$B(\mathbf{f}, \mathbf{g}) = -\oint_\Gamma \mathbf{n} \cdot \mathbf{E}^0 \times K\mathbf{g} \, d\Gamma, \qquad \mathbf{g} \in \mathscr{S}. \tag{3.1}$$

System (3.1) gives rise to a finite system of linear equations whose solution determines the vector field $\mathbf{f} \in \mathscr{S}$. This approximation scheme seems to suffer from two defects. It is not clear how to obtain the approximate scattered field in Ω_0 from $\mathbf{f}$, and it is not clear how to evaluate the operator K which appears in (3.1). These defects are overcome by a judicious choice of subspace, which we now describe.

Let $x^* \in \Omega$ be given, and let (r, θ, ϕ) be a system of spherical coordinates with the origin at x^*. In Stratton [22, p. 416], a family of vector fields

$$\mathbf{m}_{emn}, \mathbf{m}_{omn}, \mathbf{n}_{emn}, \mathbf{n}_{omn}, \qquad m = 0, 1, \ldots, n, \quad n = 1, 2, \ldots. \tag{3.2}$$

is constructed which satisfy the following properties.

(i) The fields are regular for $x \neq x^*$, and, hence, are regular in Ω_0.

(ii) The fields satisfy

$$\begin{aligned} \nabla \times \mathbf{m}_{emn} &= \sqrt{\alpha_0\beta_0}\,\mathbf{n}_{emn}, & \nabla \times \mathbf{m}_{omn} &= \sqrt{\alpha_0\beta_0}\,\mathbf{n}_{omn} \\ \nabla \times \mathbf{n}_{emn} &= \sqrt{\alpha_0\beta_0}\,\mathbf{m}_{emn}, & \nabla \times \mathbf{n}_{omn} &= \sqrt{\alpha_0\beta_0}\,\mathbf{m}_{omn}. \end{aligned} \tag{3.3}$$

(iii) The fields satisfy (2.3).

Note that we must take the Hankel functions $z_n(\rho) = h_n^1(\rho)$ in Stratton's formulas to satisfy (iii).

We fix an integer $N > 0$, and let $\mathscr{E}_N$ denote the collection of vector fields (3.2) for $0 \leqslant m \leqslant n$, $1 \leqslant n \leqslant N$. Let $\mathscr{S}_N$ denote the collection of tangential vector fields on Γ which are tangential components of vector fields $\mathbf{U} \in \mathscr{E}_N$. There are $2N(N+2)$ linearly independent fields in $\mathscr{S}_N$. If $\mathbf{g} \in \mathscr{S}_N$, using (ii), we may easily calculate $K\mathbf{g}$. If $\mathbf{f} \in \mathscr{S}_N$ is the solution of (3.1), then $\mathbf{f}$ comes from a vector field $\mathbf{U}$ in $\mathscr{E}_N$; the field $\mathbf{U}$ is the desired approximate scattered field, and may be easily evaluated at points of Ω_0. We have, therefore, shown how to overcome the defects of using (3.1).

We now discuss the system of equations arising from the use of (3.1). We arrange the fields (3.2) of $\mathscr{E}_N$ in a definite order and denote them by $\mathbf{F}_\mu$, $1 \leqslant \mu \leqslant M = 2N(N+2)$. We let $\mathbf{f}_\mu = \mathbf{n} \times (\mathbf{F}_\mu|_\Gamma)$. Thus, the $\mathbf{f}_\mu$, $1 \leqslant \mu \leqslant M$, are tangential vector fields on Γ which form a basis for $\mathscr{S}_N$. Writing the desired solution $\mathbf{f}$ of (3.1) as $\mathbf{f} = \sum c_\mu \mathbf{f}_\mu$, we obtain the linear system

$$\sum_{\mu=1}^{M} c_\mu B(\mathbf{f}_\mu, \mathbf{f}_\nu) = -\oint_\Gamma \mathbf{n} \cdot \mathbf{E}^0 \times K\mathbf{f}_\nu \, d\Gamma, \qquad 1 \leqslant \nu \leqslant M. \tag{3.4}$$

We set $a_{\mu\nu} = B(\mathbf{f}_\mu, \mathbf{f}_\nu)$, and we let $A = [a_{\mu\nu}]$ denote the coefficient matrix. We have

THEOREM. *The complex matrix A is symmetric, nonsingular, has nonsingular principal minors, and admits a factorization $A = LU$.*

Proof. The symmetry of A follows from Lemma 1. The nonsingularity of A, and of the principal minors of A, follows from Lemma 2. The existence of the decomposition then follows from the arguments of [23, Theorem 3.1]. (Note, however, that A is complex and symmetric, not Hermitian.)

Remark 1. By Lemma 1, the right-hand side of (3.1) is $-B(\mathbf{E}^0, \mathbf{g}) = -B(\mathbf{g}, \mathbf{E}^0) = -\int_\Gamma \mathbf{n} \cdot \mathbf{g} \times K\mathbf{E}^0$. Since the incident wave does not satisfy (2.3), however, $K\mathbf{E}^0$ cannot be easily calculated, so this representation is not useful.

Remark 2. An error analysis for the method proposed here will appear in a forthcoming paper.

4. Computer Implementation

Two computer programs, PCISH (Perfect Conduction In Spherical Harmonics) and PSYM (an axisymmetric version of PCISH), have been developed to implement the numerical scheme described in Section 3. PCISH is the more general of the two since it is capable of computing the electric (or magnetic) field resulting from the scattering of an arbitrary electromagnetic wave $\mathbf{E}^0, \mathbf{H}^0$ by a simply connected, perfectly conducting body Ω of arbitrary shape. PSYM is an offshoot of PCISH which is designed to handle the specific class of problems in which the boundary of Ω is a surface of revolution and the incident wave $\mathbf{E}^0, \mathbf{H}^0$ is a plane wave propagating along the axis of symmetry.

In PCISH, it is assumed that the boundary of Ω can be suitably approximated by a closed surface which is the union of a number of quadrilaterals. The vertices of the quadrilaterals and information giving the assignments of vertices to quadrilaterals comprise a major portion of the input. Other input parameters are the frequency, orientation, and shape of the incident wave, the center and maximum order N of the subspace $\mathscr{C}_N$, the quadrature order, and the coordinates of the points in space at which the scattered field is to be calculated. The input to PSYM is similar (except for specifying the incident wave) but is much simpler since the user need only supply a suitable number of points lying on a curve $\mathbf{G}$ which generates the boundary of Ω by rotation about an axis of symmetry. The actual rotation of $\mathbf{G}$ is carried out implicitly by the program.

Most of the computations performed by PCISH and PSYM center around the evaluation of the coefficient matrix and the right-hand side of the linear system (3.4). In view of the theorem, both PCISH and PSYM use Gauss elimination without pivoting to solve the matrix equation.

As for the evaluation of the coefficient matrix and right-hand side, PCISH performs a two-dimensional quadrature (based on the trapezoidal rule) over each quadrilaterial of the surface representing the boundary of Ω. PSYM, on the other hand, performs only a one-dimensional quadrature (also based on the trapezoidal rule) over a piecewise-linear arc approximating $\mathbf{G}$, which is formed by joining the given input points on $\mathbf{G}$ by line segments. The reason that only a one-dimensional quadrature is necessary is that in the case in which Γ is a surface of revolution, the quadrature in the ϕ direction has been carried out exactly by hand (requiring only the integration of some trigonometric polynomials) and the corresponding formulas placed in the program.

The assumption that the boundary of Ω is a surface of revolution has another large advantage in that, due to certain symmetries which exist in this case, many of the matrix entries are zero. Moreover, if the incident wave $\mathbf{E}^0, \mathbf{H}^0$ is assumed to be a plane wave propagating along the axis of symmetry, certain of the right-hand side entries also become zero, and the original $2N(N+2)$-dimensional system reduces to a $2N$-dimensional system, where N is again the maximum order of the subspace $\mathscr{C}_N$. Instead of using all of the vector fields (3.2), with $n = 1,..., N$, we may take subspaces generated only by the vector fields

$$\mathbf{m}_{o1n}, \mathbf{n}_{e1n}, \qquad n = 1, 2, \ldots, N. \tag{4.1}$$

The advantage of such a reduction is obvious, especially for large values of N.

In order to evaluate the integrand in the bilinear form at each quadrature point, one needs to be able to calculate the Hankel functions h_n^1 and the associated Legendre polynomials P_n^m used in the definition of the fields of (3.2). The Hankel functions $h_n^1 = j_n + iy_n$ are computed at a given point $\rho > 0$ using a backward recursion formula to calculate j_n, the nth-order spherical Bessel function of the first kind, and a forward recursion formula to calculate y_n, the nth-order spherical Bessel function of the second kind. One also needs certain derivatives of the h_n^1 and for these, we employ [24, (10.1.21)]. The Legendre polynomials and their derivatives are calculated using the recursion formulas [24, (8.5.3) and (8.5.4)].

Once the linear system has been solved by Gauss elimination, the approximate scattered field $\mathbf{E}^1$ is assembled and may be evaluated at any given point x in space. The relevant quantity for many applications is the *radar cross section* (RCS) defined by

$$\sigma(x) = \frac{4\pi R^2 |\mathbf{E}^1(x)|^2}{|\mathbf{E}^0(x)|^2}, \tag{4.2}$$

where $\mathbf{E}^0$, $\mathbf{H}^0$ is the incident wave and R is the distance of the point x from the origin x^*. To calculate the scattered field at infinity (i.e., the farfield) one takes the limit of (4.2) as $R \to \infty$. Computationally, this is achieved by using an asymptotic form of the Hankel functions when the scattered field $\mathbf{E}^1$ is assembled from the solution of the linear system.

5. Numerical Results

Both PCISH and PSYM have been developed and executed on the CDC 6500 computer located at NSWC/White Oak. These programs have been applied to a number of relatively simple model problems in order to test the program capabilities and to compare computed results with those found in the literature. Since all of the problems attempted to date have been axially symmetric, the more specialized program PSYM was used to obtain the numerical results presented below.

As mentioned in Section 4, the maximum order N and center x^* of the subspace $\mathscr{E}_N$ and the quadrature order are input parameters. In order to verify our results for a given body Ω, we manipulate these quantities in the following way: We first determine the proper value for N by successively increasing this parameter until the RCS stabilizes to some predetermined number of significant digits. However, since an increase in N causes an increase in the oscillation of the associated Legendre polynomials used in their definition, we must simultaneously increase the quadrature

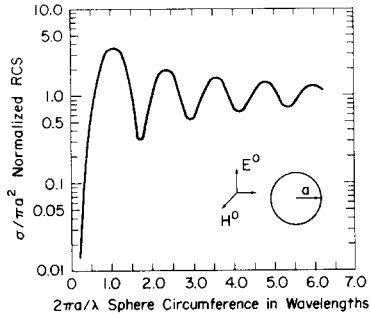

FIG. 1. Backscattering of a plane wave by a sphere.

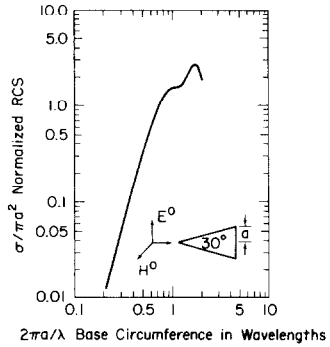

FIG. 2. Backscattering of a plane wave by a right circular cone.

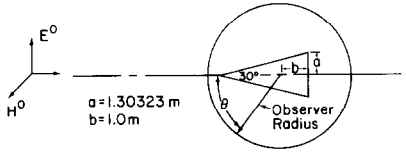

FIG. 3. Near field scattering geometry.

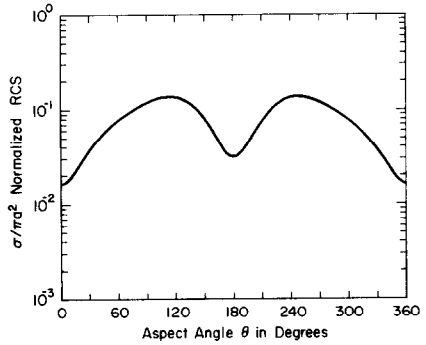

FIG. 4. Frequency: 10 MHz, observer radius: 4 m.

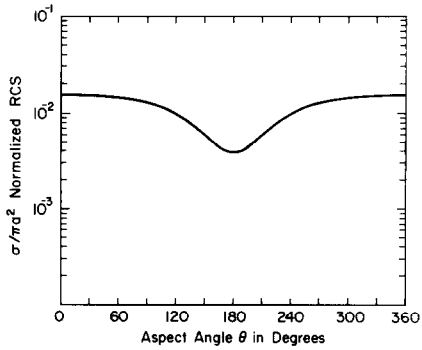

FIG. 5. Frequency: 10 MHz, observer radius: 10 m.

order to insure an accurate evaluation of the matrix system (3.4). For a given value of N, the quadrature order is increased until the RCS again stabilizes to a predetermined tolerance. Once these quantities have been determined, we perform a consistency check by varying the subspace center x^*. Since this should in theory have no effect on the farfield profile, this seems to be a fairly rigorous test of the computed values. Obviously, for each new body Ω this process involves a number of program runs. Our experience has shown that the value of N required to obtain a given number of significant digits depends on the scattering body. Thus, for a particular problem, if too many significant digits are required, the cost may become prohibitive. It is hoped that a further investigation of our method will lead to a better understanding of the interplay among these various quantities and their relationship to the other relevant parameters of the problem.

In the case in which Ω is a perfectly conducting sphere, an exact solution is known and can be found in many classical texts on electromagnetic theory (see e.g., [22]). PSYM produced extremely good farfield backscattering results in this case, even well

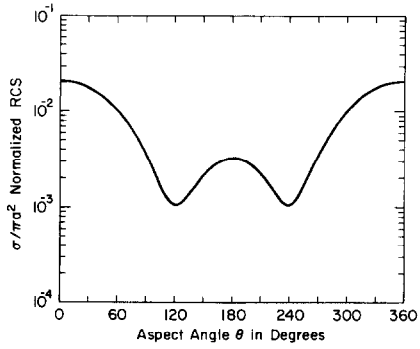

FIG. 6. Frequency: 10 MHz, farfield.

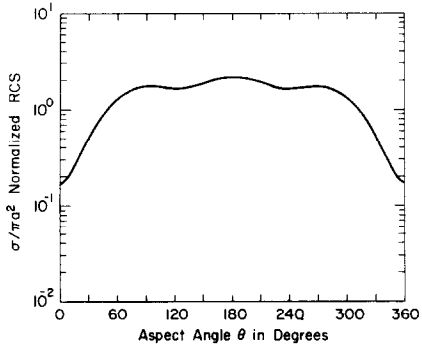

FIG. 7. Frequency: 50 MHz, observer radius: 4 m.

into the resonance region (where the wave length is of the same order of magnitude as a characteristic dimension of Ω), as can be seen by comparing Fig. 1 with a similar plot obtained from the exact solution (see [25, p. 148]). A value of $N = 8$ was used to obtain this data, and two significant digits of accuracy were obtained for the RCS.

The case in which Ω is a right circular cone is also studied in the literature [25]. For our tests, we used a right circular cone circumscribed about a 1 m sphere with a 15° half-angle at the vertex. The plane wave $\mathbf{E}^0$, $\mathbf{H}^0$ was incident upon the vertex and propagated along the axis of symmetry. Fig. 2 shows the farfield backscattering results produced by PSYM in the Rayleigh region (where the wave length is large in comparison to a characteristic dimension of Ω) and extending a short way into the resonance region. These results seem to agree with other computed and experimental results reported in [25] (in particular see figure [25, p. 392]). Using the consistency checks described above, the value $N = 30$ was found to be satisfactory, and was used to obtain the information for Fig. 2. Two significant digits of accuracy were obtained for the RCS.

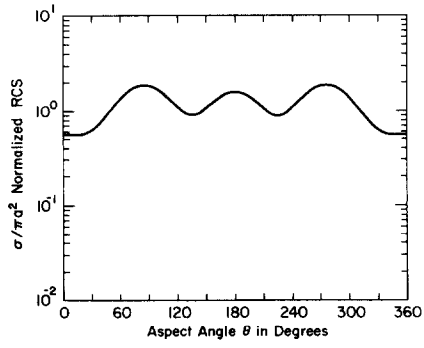

FIG. 8. Frequency: 50 MHz, farfield.

For some applications, it is desired to calculate the scattered field close to the body Ω. This presents no difficulties for PSYM, and Figs. 4–8 display some partial results in this direction for the same conical scatterer as described above. For each of the frequencies 10 and 50 MHz (corresponding to $2\pi a/\lambda = 0.273$ and 1.365, respectively, in Fig. 2), the scattered field was evaluated on circles of various radii lying in the plane parallel to $\mathbf{E}^0$ which contains the axis of symmetry of the cone (see Fig. 3). In order to obtain a farfield profile, calculations were also made on a circle, concentric with the other circles, which had an effectively infinite radius. The frequency 50 MHz is in the resonance region, and the frequency 10 MHz is on the border between the Rayleigh and resonance regions (see [26]).

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